

The closed string spectrum of $SU(N)$ gauge theories in $2 + 1$ dimensions

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Abstract

We use lattice techniques to study the closed-string spectrum of $SU(N)$ gauge theories in $2 + 1$ dimensions. We calculate the energies of the lowest lying ~ 30 states for strings with lengths between $l \sim 0.45$ fm and $l \sim 3$ fm, and compare to different theoretical predictions. We obtain unambiguous evidence that the closed-strings are in the universality class of the Nambu-Goto free bosonic string. Moreover, we clearly see that our data can be described by a covariant string theory with a small/moderate correction down to very short distance scales, and possibly on all distance scales at large- N .

I. INTRODUCTION

The generation of electric flux tubes in confining gauge theories is a basic phenomena that characterises the vacuum of these theories. In this paper we study the energy spectrum of these flux tubes using lattice techniques. We are mainly motivated by the following two reasons.

Firstly, a flux-tube whose length l is much larger than its width, is expected to be described by an effective low energy string action S_{eff} . Establishing what is the structure of S_{eff} is of fundamental interest, and can be done by studying the energy spectrum of the flux-tube. More precisely, while the energy levels E_n of a string with tension σ obey

$$E_n(l) \stackrel{l \rightarrow \infty}{\simeq} \sigma l, \quad (1.1)$$

for finite l there are corrections to Eq. (1.1) that reflect various properties of S_{eff} . In particular, the universal coefficient of the $O(1/l)$ ‘Luscher-term’ is determined by the number of the massless degrees of freedom propagating on the worldsheet of the string, thus revealing the IR universality class of S_{eff} [1]. In contrast, other details of the spectrum, like its degeneracies and the form of the subleading contributions to the Luscher term, are in general not universal and depend on the particulars of the non-renormalizable terms in S_{eff} . Consequently, by studying the l -dependence of the string spectrum, one can directly learn about S_{eff} , and about the length l_{string} above which S_{eff} begins to be a good description and a string is formed.

Our interest in the flux-tube spectrum is also driven by the following more practical reason. An essential step in any lattice study is to calculate the lattice spacing a in physical units. This is sometimes done by extracting the dimensionless combination $a\sqrt{\sigma}$ from a flux-tube’s ground state energy given in lattice units $aE_{n=0}$. Consequently, this requires that we know how E_0 depends on σ and l . To date, this dependence was approximated by correcting Eq. (1.1) with *just* the Luscher term. For the purpose of high precision lattice studies, however, this approximation can become insufficient and it is imperative to know what are the subleading corrections to Eq. (1.1) that go beyond the Luscher term.

In this work we focus on the spectrum of *closed* flux-tubes in pure $SU(N)$ gauge theories in $D = 2 + 1$ dimensions. The flux-tubes that we study have lengths that range from $l \simeq 0.45\text{fm}$ to $l \simeq 3\text{fm}$, while the lattices we use have spacings that range from $\sim 0.06\text{fm}$ to

$\sim 0.2\text{fm}$, depending on the value of N and l . (Here, despite working in $2+1$ dimensions and in pure gauge theories, we choose to *define* 1fm through the convention $\sigma \equiv (440\text{ MeV})^2$.)

The gauge groups that we study have $N = 2, 3, 4, 5, 6, 8$, and here we are motivated by the special role that the large- N limit plays in the physics of confinement; From a field theory point of view, several processes that cannot be described by a simple low energy effective string theory, such as glueball-string mixing, string/anti-string mixing and deconfinement/instability of short strings, become less important with increasing N (see the discussion in Section II C 3). This is expected to make the description of the flux-tube in terms of S_{eff} better and simplify the form of the latter. From a more general string/gauge duality point of view (for example see [2, 3]), it is the large- N limit of QCD that one may hope to describe by a string theory. This makes non-perturbative information on this limit important both to guide the searches for such a dual string theory, as well as to understand it beyond the supergravity approximation.

There are two stages to our calculation which henceforth we refer to as A and B. In stage A we perform high precision measurements of the energy of the flux-tube's ground state. In particular, we aim to isolate the different corrections to Eq. (1.1) and, while controlling them, extract a value for σ . We then use the latter to compare our data to different theoretical predictions, and especially to the Nambu-Goto (NG) free string model. This stage of our work has two immediate practical implications. Firstly, it allows us to test the analytic work in [4] which predicts that in $D = 2 + 1$ the $1/l^3$ correction to E_n has a universal coefficient, being precisely that predicted by the NG model.¹ Secondly, a precise control of the corrections to Eq. (1.1) has enabled us to perform, in a companion study [5], a precise test of the Karabali-Kim-Nair prediction for the value of σ in $2 + 1$ dimensions [6].

In stage B we put our emphasis on the excited states in the spectrum and measure the energy of the lowest ~ 30 states. This allows to extend the comparison with theoretical predictions to states with more quantum numbers and a nontrivial degeneracy structure.

The study of confining flux-tubes with lattice techniques has been an active field of research for the past three decades, and we refer the reader to some recent papers [7]. These include works that vary in the selection of the gauge group, the number of space-time

¹ Note that, using the method of [8], the authors of [9] assert that this universality persists for a general value of D .

dimensions, and the boundary conditions imposed on the flux-tube. Other papers which are particularly relevant to our current study are mentioned later on in the text.

The following is the outline of this letter. We begin in Section II by describing our methodology, proceed to Section III where we discuss the theoretical expectations for the string spectrum, and move to Sections IV-VI where we present the results of the calculations. In Section VII we summarise our results and make a few remarks on their theoretical and practical implications.

II. METHODOLOGY

In this section we describe a few aspects of our methodology. We begin with the lattice construction, proceed to discuss our general strategy to extract the energy spectrum from correlation functions, and end by listing the main systematic errors and by expanding on how we control them.

A. Lattice construction

We define the gauge theory on a discretized periodic Euclidean three dimensional space-time lattice, with spacing a . The fields are $SU(N)$ matrices assigned to the links of the lattice, and the Euclidean path integral is given by

$$Z = \int DU \exp(-\beta S_W). \quad (2.1)$$

Here β is the dimensionless lattice coupling, and for our action is related to the dimensionful coupling g^2 by

$$\lim_{a \rightarrow 0} \beta = \frac{2N}{ag^2}. \quad (2.2)$$

In the large- N limit, the 't Hooft coupling $\lambda = g^2 N$ is kept fixed, and so we must scale $\beta = 2N^2/\lambda \propto N^2$ in order to keep the lattice spacing fixed (up to $O(1/N^2)$ corrections). The action we choose to use is the standard Wilson action

$$S_W = \sum_P \left[1 - \frac{1}{N} \text{ReTr } U_P \right], \quad (2.3)$$

where P is a lattice plaquette index, and U_P is the plaquette variable obtained by multiplying link variables along the circumference of a fundamental plaquette. We calculate observables

by performing Monte-Carlo simulations of Eq. (2.1), in which we use Cabibbo-Marinari updates of the link matrices with a mixture of Kennedy-Pendelton heat bath and over-relaxation steps for all the $SU(2)$ subgroups of $SU(N)$.

In stage A we have studied $N = 2, 3, 4, 5, 6, 8$ with three lattice spacings, $a \simeq 0.06, 0.08, 0.11$ fm. In stage B we studied $SU(3)$ with $a \simeq 0.04, 0.08$ fm, and $SU(6)$ with $a \simeq 0.08$ fm. The string lengths l in both stages ranged between ~ 0.45 fm and ~ 3 fm, depending on the values of N and a . For more details on the lattice parameters of our field configurations see Tables I-II.

B. General strategy

Since we are mainly interested in the way the flux-tube spectrum reflects the properties of a standard low energy string theory of the NG type, we restrict ourselves to the spectrum of *closed* flux-tubes. By this restriction we avoid a class of short-distance contributions to the energies, such as the Coulomb interaction between sources, that cannot be easily accommodated in an effective string theory (see the discussion in Section II C 3).

We calculate the energies of flux tubes that are closed around a spatial torus. We do so from the correlators of suitably smeared Polyakov loops that wind around one of the spatial tori and that have vanishing transverse momentum. This is a standard technique [10, 11] with the smearing/blocking designed to enhance the projection of our operators onto the physical states. (We use a scheme that is the obvious dimensional reduction of the one in [12].) We classify our operators using the following quantum numbers

1. $P = \pm$: Parity in the direction transverse to Euclidean time and to the string contour.
2. $q = 0, \pm 1, \pm 2, \dots$: The momentum, in units of $2\pi/l$, along the string.

For each combination of these quanta we construct the full correlation matrix over our space of loop operators, and use it to obtain best estimates for the string states using a variational method applied to the transfer matrix $\hat{T} = e^{-aH}$ – again a standard technique [10, 11, 12, 14].

The size N_o of our $N_o \times N_o$ correlation matrices depend on the states we are interested in. In stage A, where we focus on the lowest state, our correlations are built from Polyakov loops that wind once around the torus in straight paths and in all possible blocking levels. Consequently, this means that we restrict ourselves to states with $q = 0$ and $P = +$, and this

provides us with $N_o = 3 - 5$ states. In stage B we include Polyakov loops with many different paths (wave-like paths, pulse-like paths, etc.). This increases our number of operators to $N_o \simeq 80 - 200$ and allows us to probe states with negative P and nonzero q . Finally, an extension of our calculation for the ground state to $w > 1$ will be reported in [15].

Once we obtain the string energies for different values of l , we fit the ground state energy in powers of $1/l$ and obtain an estimate for the function $E_0(l, \sigma)$. With this empirical formula we extract the string tension in lattice units, $a^2\sigma$, for each data set. Substituting the results in different theoretical predictions for the spectrum, we conclude by comparing the latter with our data.

C. Systematic errors

We control several systematic errors in different stages of our study. These include the way we extract the string energies from correlation functions, the way our results approach the infinite volume and continuum limits, and the way one may interpret the results for *small* values of l as reflecting an effective string picture. In the next three sub-sections we expand on each systematic error, and on the way that we control them.

1. Extracting string energies from correlation functions

The output of the variational technique is a set of operators that couples “best” to a set of low energy states. For example, our best operator for the string’s ground state has typically an overlap $\sim 99\%$ onto that state so that the normalised ‘ground state’ correlation function satisfies

$$C(t) = (1 - |\epsilon|) \exp\{-E_0(l)t\} + |\epsilon_1| \exp\{-E_1(l)t\} + \dots \quad ; \quad \sum_i |\epsilon_i| = |\epsilon| \sim 0.01 \quad (2.4)$$

where E_0, E_1 are the ground and first excited state string energies. (Since our time-torus is finite, we use cosh fits rather than simple exponentials, although in practice we use L_t large enough for any contributions around the ‘back’ of the torus to be negligible.) To extract E_0 from this correlator one can fit with a single exponential for $t \geq t_0$, discarding $t < t_0$, and choosing t_0 to be the minimum value so that a statistically acceptable fit is obtained. This is a reasonable approach and one followed in [10, 11]. However it neglects the systematic error

arising from the fact that there is certainly some excited state contribution as demonstrated, for example, by the fact that one cannot obtain a good fit with a single exponential from $t = 0$. To control this systematic error we also perform fits with two exponentials, with a fixed mass E^* for the excited state, resulting in a mass $E_0(E^*)$ for the ground state. Typically $E_0(E^*)$ is smallest when E^* is as small as possible, i.e. $E^* = E_1$, and is largest when $E^* = \infty$, i.e. effectively a single-exponential fit. So the true value typically satisfies:

$$E_0(E_1) \leq E_0^{\text{true}} \leq E_0(\infty). \quad (2.5)$$

From here on, we refer to the single-cosh fitting procedure by ‘S’, and to the double-cosh fitting procedure, by ‘D’. This particular systematic error becomes more important as the string length l increases because the overlap of our lattice operators typically decreases with increasing l , and the excited state energy approaches the ground state energy. In practice, we find that as long as $l \lesssim 5/\sqrt{\sigma}$ one can neglect this systematic error at the level of our current statistical errors. Consequently we present results from the ‘D’ fitting procedure only when $l > 5/\sqrt{\sigma}$.

2. Finite volume and discretisation effects

To avoid finite volume effects in the calculation of the closed string spectrum we follow [13] and increase the transverse and temporal extents of the torus, when we decrease the length of the string l . We performed explicit finite volume checks for a restricted set of parameters, and found that the transverse volumes used in our study are large enough to avoid any observable effects at the level of our statistical accuracy. To check for finite lattice spacing effects we perform several of our calculations with different lattice spacings.

3. A string interpretation of the flux-tube spectrum for short lengths ?

The energy spectrum of confining flux-tube is expected to be much more complicated than that of a simple effective string and to approach the latter only at large l . Therefore, in an ideal calculation one would study the $1/l$ terms in the string energies only for $l \gg 1/\sqrt{\sigma}$. In practice, however, the $1/l$ terms are numerically small and a useful measurement at large l would require an unrealistically large statistical sample. Instead, we study strings with

$1/\sqrt{\sigma} \lesssim l \lesssim 6/\sqrt{\sigma}$ for which the corrections are not negligible and can be reasonably fitted. This procedure is not free of ambiguities since the short- l spectrum may be sensitive to phenomena not accommodated in an effective string theory, but as we explain below, it is unlikely that this ambiguity is significant in our calculation.

The first obvious source of short-distance “contamination” in studies of flux-tube spectra is the presence of a Coulomb interaction between the static charges at the ends of *open* flux-tubes. This can be a significant portion of the total energy when the flux tube is short, so it is not clear whether the deviations from the infinite length limit seen in the open channel (for example see [17] and reference within) are due to this Coulomb interaction or whether they reflect string interactions. Our calculation, however, is free of this contamination simply because we study *closed* flux-tubes and by construction these do not have static charges and a Coulomb interaction.

A different type of non-stringy phenomenon that does occur in the closed channel is the deconfinement transition. In $2+1$ dimensions this finite temperature transition takes place at a temperature $T = T_d \simeq 0.9/\sqrt{\sigma}$ [18]. By identifying the length of the compact direction, l , with the inverse temperature T^{-1} , this means that our calculation necessarily breaks down when $l < 1/T_d \simeq 1.11/\sqrt{\sigma}$, but also that interpreting the flux-tube energies as coming from the dynamics of S_{eff} may be questionable in the confined phase, when $l \simeq (1/T_d)^+$. This is particularly true for $N = 2, 3$ when the transition is second order and the $l \rightarrow (1/T_d)^+$ behaviour of $E_0(l)$ will be governed by appropriate critical exponents, and may also occur for $N = 4, 5$, where the first order transition is relatively weak. In contrast, for $N \geq 6$, the transition is strongly first order and it is quite possible that a confining string description exists even when $l \simeq (1/T_d)^+$.²

Another short-distance phenomenon involves glueballs : by decreasing l the energy needed to excite the string grows (see Section III) and if we take the spectrum of the free bosonic string prediction as a guide, then the threshold for the first excited state to emit the lightest 0^{++} glueball and decay to the ground state is reached when $l \simeq 1.53/\sqrt{\sigma}$. In practice, however, since the amplitudes of these mixing processes are subleading in $1/N^2$, they may be suppressed even for $SU(3)$.

² As an extreme example, note that for very large values of N and in $3+1$ dimensions, it is possible to study confining flux-tubes even *below* $1/T_d$ [19].

To conclude, the short-distance non-stringy phenomena that may contaminate the string picture interpretation of the closed flux-tube spectrum, go away at large- N . Hence, by performing our calculations for increasingly large values of N we explicitly check how large are these effects and thereby control them.

III. THEORETICAL EXPECTATIONS FOR $E_n(L)$

Let us now discuss the theoretical predictions to which we compare the measured flux-tube spectrum.

A. The spectrum of the Nambu-Goto model

The action of the NG model [20] is the area of the worldsheet swept by the propagation of the string. Due to the Weyl anomaly this model is quantum-mechanically consistent only in the critical dimension $D = 26$ (see for example [21]), but since this anomaly is suppressed for long strings [22] it can still be considered as an effective low energy model.

The single string states can be characterized by the number of times w that the string winds around the torus. The spectrum in each case corresponds to the transverse oscillations of the worldsheet that correspond to movers that travel clockwise and anti-clockwise along the string. Thus the string states are characterised by w , by the occupation number $n_{L(R)}(k)$ of left(right) movers that carry energy k , and also by the centre of mass momentum $\vec{p}_{\text{c.m.}}$. By projecting to zero transverse momentum $(\vec{p}_{\text{c.m.}})_\perp$ we are left only with the momentum along the string axis which is quantized in units of $2\pi q/l$ with $q = 0, \pm 1, \pm 2, \dots$ for a string of length l . These quanta are not independent of $n_{L,R}$ and obey the level matching constraint³

$$N_L - N_R = qw, \quad (3.1)$$

where $N_{L(R)}$ enumerates the momentum contribution of the left(right) movers in a certain state as follows

$$N_L = \sum_{k>0} \sum_{n_L(k)>0} n_L(k) k, \quad N_R = \sum_{k'>0} \sum_{n_R(k')>0} n_R(k') k'. \quad (3.2)$$

³ This condition constraints the physical states to be invariant under the gauged diff-invariance of the action [21], and is effectively momentum conservation.

It is customary to characterise the string states as irreducible representations of the $SO(D-2)$ symmetry that rotates the spatial directions transverse to the string axis. In our $D = 2 + 1$ dimensional case, this group becomes the transverse parity P and acts by assigning a minus sign for each mover on the worldsheet. As a result the string states are eigenvectors of P with eigenvalues

$$P = (-1)^{\sum_{i=1} n_L(k_i) + \sum_{j=1} n_R(k'_j)}. \quad (3.3)$$

Finally, the energy of a closed-string state with the above quanta is (here we write it for a general number of spacetime dimensions D)

$$(E_{N_L, N_R, q, w})^2 = (\sigma l w)^2 + 8\pi\sigma \left(\frac{N_L + N_R}{2} - \frac{D-2}{24} \right) + \left(\frac{2\pi q}{l} \right)^2. \quad (3.4)$$

B. Effective string theories

Since in $2 + 1$ dimensions the NG string is at best an effective low-energy string theory, it makes sense to generalise it and write the most general form of an effective string action S_{eff} consistent with the symmetries of the flux-tube system. This was done some time ago for the $w = 1$ and $q = 0$ in [1] and the spectrum obtained for a general number of space-time dimensions D was

$$E_n = \sigma l + \frac{4\pi}{l} \left(n - \frac{D-2}{24} \right) + O(1/l^2), \quad (3.5)$$

with $n = 0, 1, 2, \dots$. Here the second term on the right hand side of Eq. (3.5) is known as the Luscher term and is expected to be universal and independent of the particulars of IR-irrelevant interactions of the low energy effective string theories. Indeed, it can be easily verified that the NG model obeys this universality by expanding the square-root of Eq. (3.4) to leading order in $1/l$.

The work [1] was more recently extended in [4], where the authors established and used a certain open-closed string duality of the effective string theory. Using this duality they showed that for any number of spacetime dimensions the $O(1/l^2)$ is absent from Eq. (3.5), and that in $D = 2 + 1$ the $O(1/l^3)$ has a universal coefficient. Consequently, in $2 + 1$ dimensions, Eq. (3.5) is extended to

$$E_n = \sigma l + \frac{4\pi}{l} \left(n - \frac{1}{24} \right) - \frac{8\pi^2}{\sigma l^3} \left(n - \frac{1}{24} \right)^2 + O(1/l^4). \quad (3.6)$$

In the context of the covariant string description such as that of Section III A, Eq. (3.6) implies that

$$E_n = \sqrt{(\sigma l)^2 + 8\pi\sigma \left(n - \frac{1}{24}\right) + O(1/l^3)}, \quad (3.7)$$

which is a particularly convenient form since the two first terms under the square root are the prediction of the NG model for the energy squared, that we find to be a very good approximation (see below). A different approach that also leads to similar conclusions is the Polchinski-Strominger effective string theory [8]. While it originally yielded Eq. (3.5), it was used recently in [9] to extend the analysis to higher powers of $1/l$ and leads to the same conclusions as [4], but for all values of D .

Motivated by these recent developments, we take our main fitting ansatz for the spectrum to be

$$E_{\text{fit}} = \sqrt{E_{NG}^2 - \sigma \frac{C_p}{(l\sqrt{\sigma})^p}} \quad ; \quad p \geq 3, \quad (3.8)$$

where E_{NG}^2 is the NG prediction given by Eq. (3.4) and where C_p are dimensionless coefficients that in general can depend on the quantum numbers of the state.

Let us pause and make the following comment about the relation between Eq. (3.6) and Eq. (3.7). Consider performing a large- l expansion of the square-root in Eq. (3.7). The result would include not only Eq. (3.6), but also many additional terms that begin at $O(1/l^5)$, and that come from the expansion of the second term under the square-root. To see whether these extra terms are important, we can use our data and compare its deviations from Eq. (3.6) to its deviations from Eq. (3.7). If we find that the latter are systematically smaller, this would then reflect the naturalness of adding a correction term in a covariant way (correcting E_n^2 as in Eq. (3.7) and Eq. (3.8)) rather than correcting the energy E_n itself (as Eq. (3.6) does). From a field-theoretical point of view, such as the one in [4], it seems that it would be hard to get a prediction of the form of Eq. (3.7) since it can be viewed as an implicit resummation of an infinite series of powers of the Luscher term.

IV. A PRECISE MEASUREMENT OF THE GROUND STATE ENERGY $E_0(\sigma, l)$

We performed high precision calculations of the dependence of the ground state energy on the length of the string. The calculations were done for the numbers of colors $N = 2, 3, 4, 5, 6, 8$. The lengths of the strings were restricted to obey $l > 1/T_d$, with the

deconfinement transition temperature T_d given in [18]. As examples, we present the results for $N = 3$ and $N = 6$ with $\beta = 14.7172$ and $\beta = 90.00$ respectively, in Fig. 1.

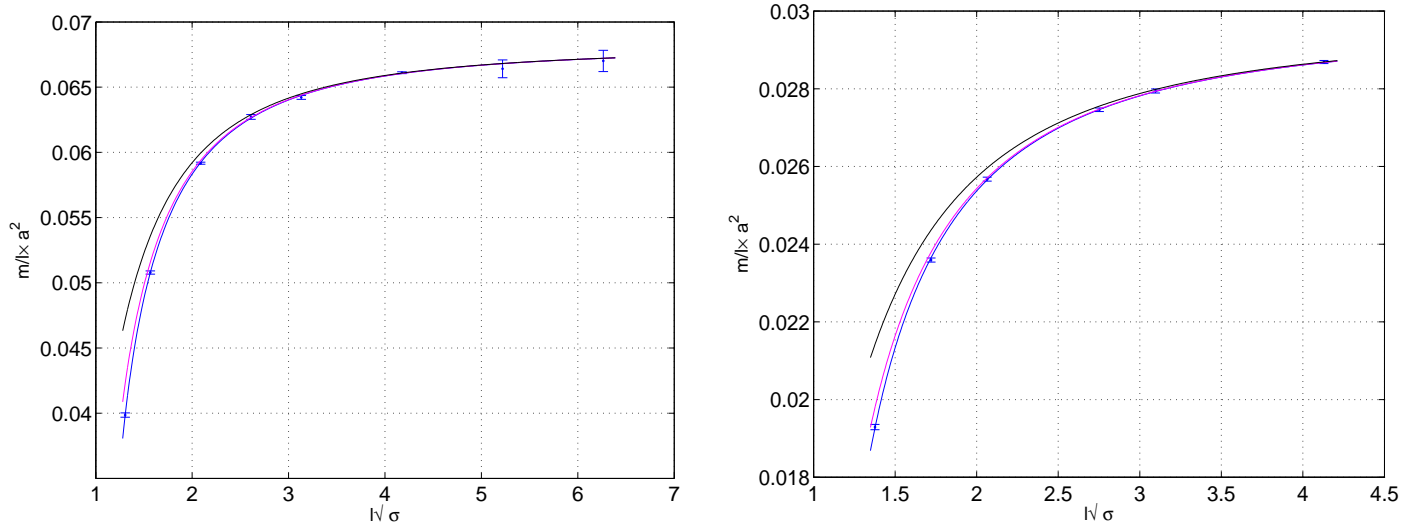


FIG. 1: The ground state energy per unit length, in lattice units, vs. the string length for $SU(3)$ and $\beta = 14.7172$ (left panel) and for $SU(6)$ and $\beta = 90.00$ (right panel). All the data points are the result of the single exponential fits, except when $l > 5/\sqrt{\sigma}$, where we use the results from double exponential fits (see Section II C 1). The blue lines are the results of using the fitting ansatz Eq. (3.8). Using the values we obtain for the string tensions we plot the NG predictions of Eq. (3.4) (magenta lines) and of the Luscher formula in Eq. (3.5) (black lines).

Fitting the data with Eq. (3.8) and $p = 3$ we obtain acceptable fits only when we use the ‘D’ data for strings with $l \gtrsim 5/\sqrt{\sigma}$. This indicates that for long strings, for which the first excited energy is relatively close to the ground state energy, the control over contamination of the excited states is crucial. The results for the other gauge groups and lattice spacings are similar to the $SU(6)$ results, and we present the parameters and goodness of fits of the form in Eq. (3.8) with $p = 3$ in Table I.

From the table one can also see that the N -dependence and the lattice spacing dependence of the correction C_3 is moderate/small. We have also performed fits with $p = 4$, and obtained comparable χ^2 , which means that with the current data for the ground state, we cannot unambiguously determine the power of the correction term in Eq. (3.8).

As Fig. 1 demonstrates, the Luscher term formula and the NG formula are indistinguishable from each other and agree with the fit at large values of l . The situation at small

Configuration details	$a^2\sigma$	C_3	Confidence level
$SU(2), \beta = 5.6$	0.074636(25)	0.0100(41)	53%
$SU(3), \beta = 14.7172$	0.068127(47)	0.1633(112)	87%
$SU(4), \beta = 28.00$	0.063346(73)	0.2943(282)	99%
$SU(4), \beta = 50.00$	0.017184(14)	0.1886(338)	10%
$SU(5), \beta = 80.00$	0.016874(12)	0.0554(139)	68%
$SU(6), \beta = 59.40$	0.077460(81)	0.1098(169)	16%
$SU(6), \beta = 90.00$	0.029601(23)	0.1163(159)	88%
$SU(8), \beta = 108.00$	0.075351(141)	0.1865(575)	68%
$SU(8), \beta = 192.00$	0.020200(24)	0.0864(109)	–

TABLE I: The lattice parameters β and $a^2\sigma$ and C_3 in the fit Eq. (3.8). The fits were performed for the S data sets except for strings of length $l > 5/\sqrt{\sigma}$, where we used the D type of data (Single and Double exponential fitting of the correlation functions - see Section II C 1). For $SU(8)$ and $\beta = 192.00$ our confidence level is small, and comes from a large scatter of the data points around the fit. This indicates that our errors are probably underestimated in this case.

values of l , however, is completely different. There, the Luscher term is clearly insufficient to describe the data, indicating the importance of the subleading corrections to Eq. (3.5) for $l \lesssim 3/\sqrt{\sigma} \simeq 1.35$ fm. In contrast, the NG prediction is very good, even at the lowest value of l . This remarkable fact also explains why we choose Eq. (3.8) to be our fitting ansatz, rather than using a fit which assumes that the energy itself is a power series in $1/l$.

Finally, we perform the following two type of fits

$$\text{Fit 1.} \quad E_0(l, \sigma) = \sigma l - \frac{\pi}{6} \times C_{\text{eff}}^{(1)}, \quad (4.1)$$

$$\text{Fit 2.} \quad E_0(l, \sigma) = \sqrt{(\sigma l)^2 - \frac{\pi\sigma}{3} \times C_{\text{eff}}^{(2)}}, \quad (4.2)$$

to pairs of adjacent values of l in plots of the type of Fig. 1.

The effective central charge $C_{\text{eff}}(l)$ in both type of fits is expected to approach a value of unity for large values of l , but to deviate from 1 when l decreases because of the higher order $O(1/l^3)$ terms. The fit in Eq. (4.1) is the one usually performed (see for example [23]) and is the analog of the effective central charge fits performed in the open channel [4, 17, 24]. In contrast, the ansatz in Eq. (4.2) assumes that most of $C_{\text{eff}}^{(1)}$'s deviation from 1 comes from

the $1/l$ terms that are accommodated in the NG prediction. Indeed, this assumption is confirmed by the data and $C_{\text{eff}}^{(2)}$ approaches 1 much faster than $C_{\text{eff}}^{(1)}$, and we present both, for the ‘S’ data sets of $SU(3)$ with $\beta = 14.7172$ and of $SU(6)$ with $\beta = 90.00$, in Fig. 2.

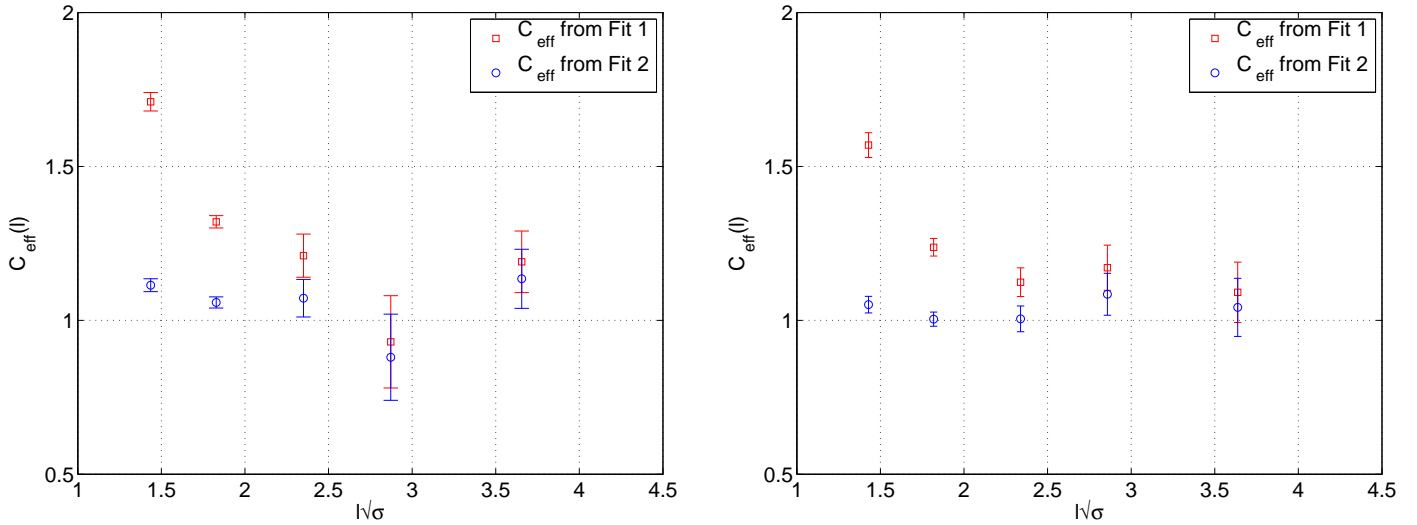


FIG. 2: The effective central charges $C_{\text{eff}}^{(1,2)}$ vs. the string length for the ‘S’ data points, for $SU(3)$ and $\beta = 14.7172$ in the left panel, and for $SU(5)$ and $\beta = 80.00$ in the right panel.

Finally, in our largest statistical sample, which was obtained for $SU(2)$, we find

$$C_{\text{eff}}^{(2)} = 0.9991(55) \quad \text{at} \quad l \simeq 0.68 \text{ fm.} \quad (4.3)$$

V. TEST OF THE NG PREDICTION FOR THE 1ST EXCITED STATE

Equipped with values for the string tensions, we are now in a position to test the parameter-free predictions of the NG model and of the Luscher term formula for the first excited states. We substitute the values of $a^2\sigma$ that appear in Table I in Eq. (3.4) and Eq. (3.5) for $w = 1, q = 0$ and $N_R = N_L = 1$ and compare with our data. We present the comparison for $SU(3)$ and $SU(6)$, and for two different lattice spacings in Fig. 3.

In contrast to the case of the ground state, both the predictions Eq. (3.5) and Eq. (3.6) are unable to describe our data for all the values of l that we study, and it seems that they will become consistent with the data only when $l \simeq (6-7)/\sqrt{\sigma} \simeq 2.7-3.2$ fm. Despite this, the NG prediction is strikingly within $\sim 5\%$ of our data for $l \gtrsim 2/\sqrt{\sigma} \simeq 0.9$ fm, and becomes consistent when $l \gtrsim 3.5/\sqrt{\sigma} \simeq 1.6$ fm. To further check how this agreement depends on N

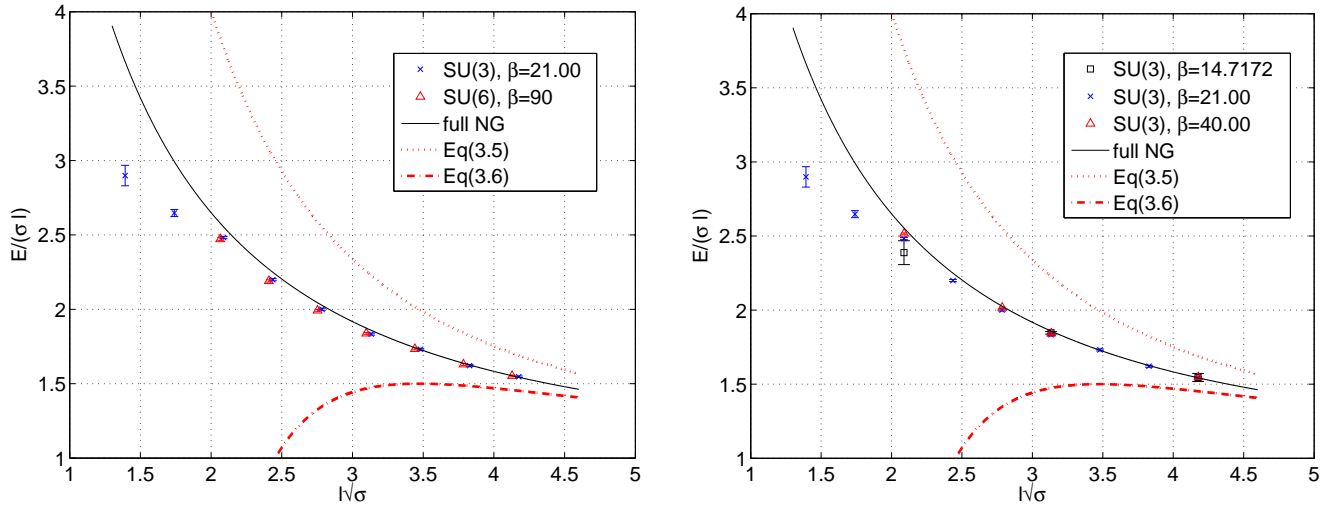


FIG. 3: The energy of the 1st excited state, divided by σl (See Tables I-II for the values of the string tensions). Left panel: Comparing $SU(3)$ and $SU(6)$ for a similar lattice spacing of $a^2\sigma \simeq 0.03$. Right panel: Comparing different lattice spacings for $SU(3)$. The black line is the NG prediction, while the dotted(dashed) lines are the NG prediction expanded to leading and next-to-leading order.

and whether it evolves with the lattice spacing a , we have repeated this analysis for gauge groups with $2 \leq N \leq 8$ and a large set of lattice spacings with $0.2 \text{ fm} \geq a \geq 0.05 \text{ fm}$, and string lengths with $l \gtrsim 3/\sqrt{\sigma}$, and this will be presented in a forthcoming publication [25]. What we find is that the dependence on both a and N is not significant.

VI. COMPARISON OF THE NG SPECTRUM WITH THE LOWEST ~ 30 STATES

In this section we present results from an extensive calculation of the lowest ~ 30 states in the spectrum. We restrict the discussion to states with $w = 1$, $q = 0, 1, 2$ and $N_{L,R} \leq 3$. We have obtained results for the third excited states with $q = 0$ and $N_L = N_R = 3$ as well, but we postpone their presentation to [25]. The results were obtained for $SU(3)$ with $\beta = 21.00, 40.00$ and for $SU(6)$ with $\beta = 90.00$. Here we have focused on strings with $1.4/\sqrt{\sigma} \lesssim l \lesssim 5.5/\sqrt{\sigma}$. All energies that we present here were obtained from single exponential fits.

We present the results in Figs. 4-5. The lines are the predictions of NG model. The string tensions used for these predictions were extracted from the ground state energies with the

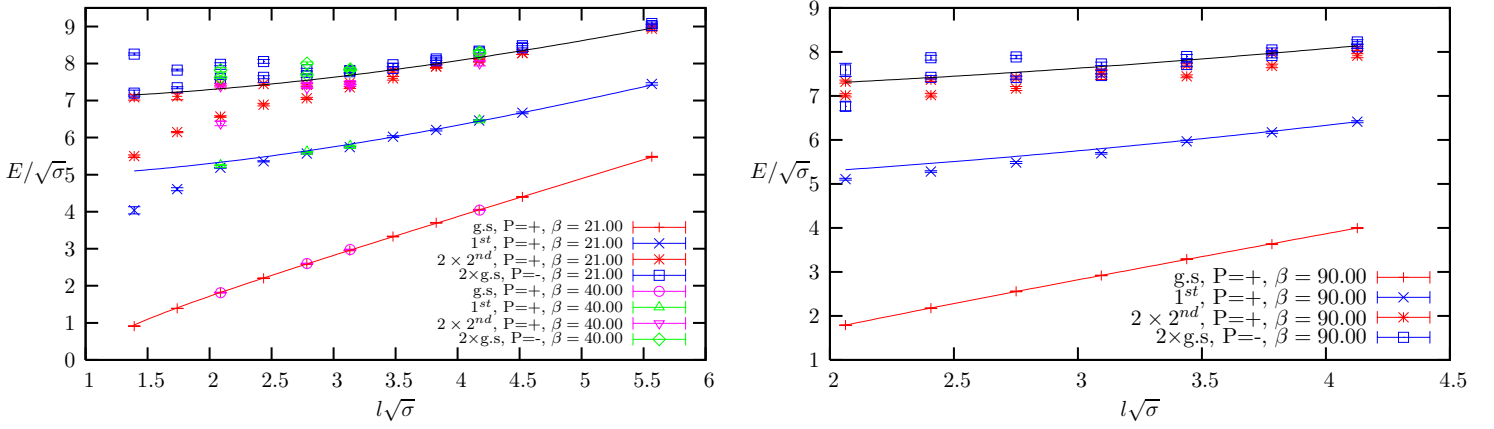


FIG. 4: The energies of the lowest 7 states in units of $\sqrt{\sigma}$ as a function of $l\sqrt{\sigma}$. The three lines are the NG predictions for the ground state (red), 1^{st} excited state (blue) and 2^{nd} excited state (black). Left panel: We present the results for the case of $SU(3)$ for two different values of β (two different lattice spacings). Right panel: Results for $SU(6)$ with $\beta = 90.000$. In both panels we denote the degeneracy of the states in the legends.

use of the fitting ansatz Eq. (3.8), and we present the fitting parameters in Table II.

Gauge group	β	$a^2\sigma$	C_3	Confidence level
$SU(3)$	21.00	0.030258(26)	0.160(21)	69%
	40.00	0.007577(13)	0.05(31)	15%
$SU(6)$	90.00	0.029559(36)	0.04(21)	88%

TABLE II: The parameters $a^2\sigma$ and C_3 in the fit Eq. (3.8), which are obtained for the S data sets. The errors on C_3 are much larger than the ones that appear in Table I because here we did not study the very short strings.

It is clearly seen that the NG predictions are very good approximations to the flux-tube spectrum and deviate from our data only at the level of $\sim 2\%$ once $l \gtrsim 4.2/\sqrt{\sigma} \simeq 1.9$ fm. At this level some of the systematic errors may be significant. Note that the degeneracy pattern predicted by the NG model is seen from our data : the second energy level is fourfold degenerate at large- l . This degeneracy includes two positive parity states and two negative parity states, and these start splitting significantly once $l \lesssim 3/\sqrt{\sigma} \simeq 1.35$ fm.

We note in passing that the $1/l$ and $1/l^3$ contributions to the energies, which were predicted using effective field-theories are clearly insufficient to describe our data, and would

presumably do so only for much longer strings. In contrast, the implicit resummation of powers of the Luscher term which appears in the covariant string expression (see discussion at the end of Section III B) is strongly supported by our data.

Next, we fitted the data for the excited states. In the case of the first excited energy level, where there is only one state per level, we used the fitting ansatz Eq. (3.8). In the case of the second excited energy level, where for each parity there are two states, we used a modified form of Eq. (3.8) and fitted the difference between the energies squared of these states to the form

$$(\delta E)^2 = \sigma \frac{C_p}{(l\sqrt{\sigma})^p}. \quad (6.1)$$

The results of these fits are presented in Table III.

To compare our data to the Luscher-Weisz prediction in [4] we momentarily assume that $p = 2$. Expanding Eq. (3.8) this assumption results in an $O(1/l^3)$ contribution to the string energy which deviates from the Luscher-Weisz prediction by an amount proportional to C_2 . In the case of the first excited state and $l > 2/\sqrt{\sigma}$, we substitute the values $C_2 \simeq 3 - 8$ from Table III, and find that this deviation is only at the level of 2% – 6%. The smallness of this deviation, and the largeness of the errors we find for p from our fits (see Table III), demonstrate that to unambiguously determine p , and test the Luscher-Weisz prediction, requires statistical errors which are at least 2-3 times smaller than the ones our current data has, and a simultaneous control of any systematic errors that may be important at this level of few percents accuracy.

Finally, we move to the nonzero longitudinal momentum q sector. As mentioned in Section III, in the NG prediction of Eq. (3.4), the number of left and right movers are constrained to obey the level matching condition Eq. (3.1). The comparison of this prediction with our data is given in Fig. 5, where we present $\sqrt{E^2/\sigma - (2\pi q/\sqrt{\sigma}l)^2}$ as a function of $l\sqrt{\sigma}$.⁴ As clearly seen the data is indeed very well described by Eq. (3.4).

VII. CONCLUSIONS

We have calculated the energy spectrum of closed strings in $SU(N)$ gauge theories in $2 + 1$ dimensions with the string length l in the range $0.45 \text{ fm} \leq l \leq 3 \text{ fm}$.

⁴ This way of presenting the data separates the data sets and eases the representation.

Level	gauge group	β	$l\sqrt{\sigma}$	p	C_p	$\chi^2/d.o.f$
1 st excited	$SU(3)$	21.00	2.1	1.8(5)	6(3)	2.4/7
	$SU(3)$	21.00	1.4	3.7(2)	37(5)	39.5/9
	$SU(3)$	40.00	2.1	1.7(7)	3(2)	2.0/2
	$SU(6)$	90.00	2.1	1.6(4)	8(3)	3.8/5
2 nd excited, $P = +$	$SU(3)$	21.00	2.1	3.0(3)	53(15)	4.2/7
	$SU(3)$	40.00	2.1	0.5(5)	4(2)	3.2/4
	$SU(6)$	90.00	2.1	0.3(3)	3(1)	3.4/5
2 nd excited, $P = -$	$SU(3)$	21.00	2.1	2.1(7)	16(9)	7.9/7
	$SU(6)$	90.00	2.1	2.5(5)	36(18)	5.5/5

TABLE III: The parameter C_p in the fit Eq. (3.8) and $\chi^2/d.o.f$, which were obtained for the S data sets (see Section II C 1). For the second excited level we fit both the positive parity ($P = +$) and negative parity ($P = -$) states.

For the ground state we have studied $2 \leq N \leq 8$ for lattice spacings $0.06 \leq a \leq 0.11$ fm, depending on N , and saw that the Nambu-Goto (NG) free bosonic string model describes our data very well even at the relatively short distances of $l \simeq 0.6 - 0.7$ fm (this was already noted in [5]). In particular, we find that the central charge c is in general consistent with 1. For example, in $SU(2)$ we see $c = 0.9991(55)$ when we extract it already at $l \simeq 0.68$ fm, while in $SU(5)$ we find $c = 1.004(23)$ at $l \simeq 0.82$ fm. This provides unambiguous evidence that the closed flux-tube is in the bosonic string universality class.

To study the excited string spectrum we constructed a basis of $\sim 80 - 200$ operators in each channel, and from a variational calculation we extracted the lowest ~ 30 states. These include states with positive and negative parity, as well as with nonzero momentum along the string direction. In general we find that the agreement with the NG prediction in Eq. (3.4) is very good, including the expected degeneracy pattern.

This agreement is in striking contrast to what we find when we simply compare to the Luscher term as in Eq. (3.5) or to the Luscher-Weisz prediction in Eq. (3.6). While for the ground states these describe our data already at $l \simeq 1.35$ fm, where they are indistinguishable from the full NG formula, for the excited states the situation is completely different. In particular, whereas the NG prediction works well for the first excited state already at $l \simeq 1.35$

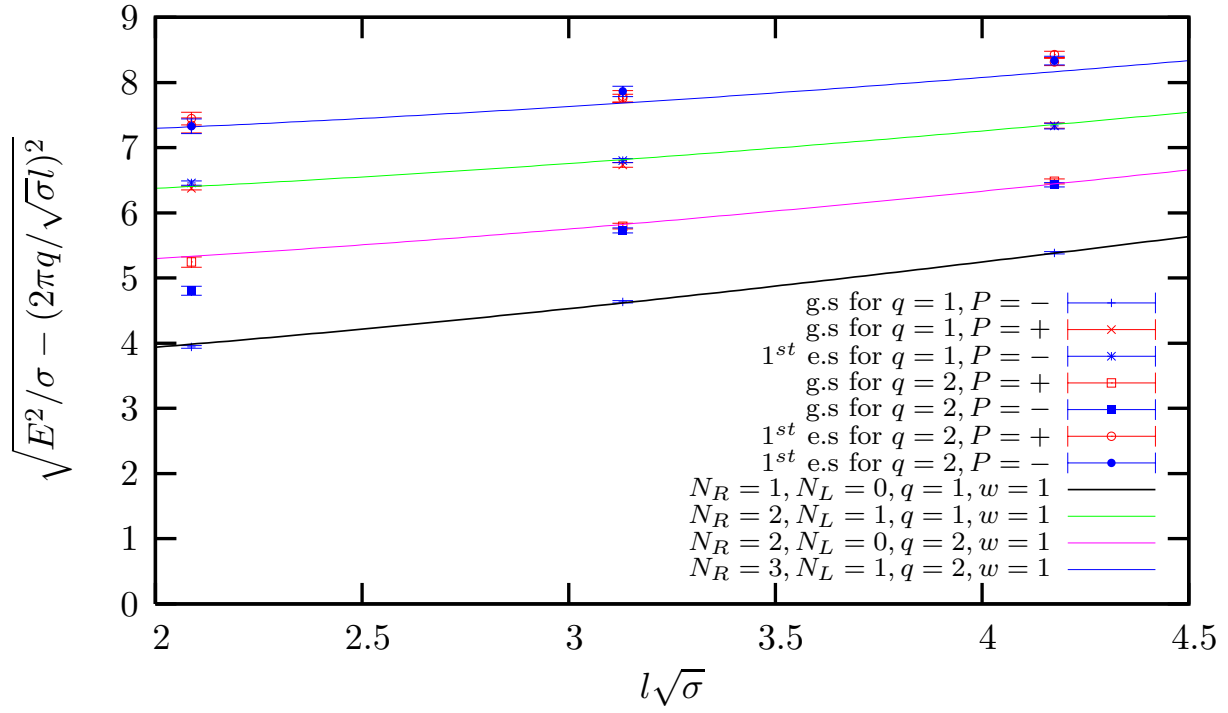


FIG. 5: $\sqrt{E^2/\sigma - (2\pi q/\sqrt{\sigma}l)^2}$ as a function of $l\sqrt{\sigma}$ for the ground state (g.s.) and excited states (e.s.) of the non-zero q sector. The four lines present the NG prediction Eq. (3.4). For $N_R = 3, N_L = 1$ and $q = 2$, the expected degeneracy is three, which agrees with our excited state data, i.e. two(one) states with positive(negative) parity in red(blue).

fm, the predictions of Eqs. (3.5) and (3.6) are still very far from the data.

This means that our results show unambiguously that the confining flux-tube can be described by a covariant string theory with small/moderate corrections, down to very short distance scales, and possibly at all distance scales at large- N .

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